Friction vs. damage: dynamic self-similar crack growth revisited

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Abstract Seismological observational studies have revealed that earthquakes exhibit dynamic self-similar crack growth constituting 50–90% of the shear wave velocity. Remarkably, the peak slip velocity defined on the crack surface is scale-invariant, even from $M1$ to $M9$ earthquakes. However, a classical self-similar crack model with a singularity does not satisfy all the observed properties above. In this chapter, we review these discrepancies and introduce friction and damage models to solve them, which have been proposed in several numerical studies. We show that velocity-dependent friction can fulfill some requirements of the observations, while slip- or time-dependent friction cannot. We finally discuss the theoretical equivalence of friction and damage model for a self-similar crack in terms of energetics, which has previously only been implied by numerical studies.

1 Introduction: applications and limitations of a classical crack model for earthquake mechanics

Earthquakes involve dynamic rupture propagation with speed of $2-3$ km/s along faults, which are shear cracks embedded in the Earth's crust. To model this faulting process mathematically, we need to define the amount of slip on faults as a function of space and time and the magnitude of earthquakes. Let $u(x,t) \in \mathbb{R}^3$ be displacement of the medium defined at position $x \in \mathbb{R}^3 \backslash \Gamma$, where t is time since the initiation of dynamic rupture and Γ is a fault surface. Then, we define slip $D(x,t)$ and slip velocity $V(x,t)$ on Γ as

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$$D(x,t) := \lim_{\varepsilon\downarrow 0} u\left(x+\varepsilon\nu^{\Gamma},t\right) - \lim_{\varepsilon\uparrow 0} u\left(x+\varepsilon\nu^{\Gamma},t\right), \tag{1}$$

$$V(x,t) := \partial_t D(x,t), \tag{2}$$

respectively, where ν^{Γ} is a unit normal vector to Γ and $D = V = 0$ is assumed for $t \leq 0$. Earthquake size is quantified by the seismic moment, M_0, originally defined as[1]

$$M_0 := \mu \int_{\Gamma} |D(x,T)|\, dx,$$

where μ is the rigidity, and T is the duration of the earthquake, i.e., the amount of time required to generate the final state of the earthquake process. By extending this definition, the time-dependent seismic moment function $M_0(t)$ can be defined as

$$M_0(t) := \mu \int_{\Gamma} |D(x,t)|\, dx. \tag{3}$$

The static M_0 is proportional to the minimum value of elastic strain energy released by the earthquake if the on-fault traction change is spatially constant; see Hirano[12] and references therein for a precise review and practical problems with this relationship. Additionally, the (seismic moment) magnitude $M_w := \frac{2}{3}(\log_{10} M_0 - 9.1)$ is widely employed.

The duration, T, can reach $\sim 10^2$–10^3 s for $M9$ earthquakes, which are the largest events on the earth, while T is at most ~ 1 and $\sim 10^{-6}$ s for $M4$ (medium-sized in nature) and $M-8$ (smallest events observed in the laboratory) earthquakes, respectively. Hence, earthquake duration varies 10^9 fold from the smallest to the largest. A question then arises: are the mechanisms of small, medium, and large earthquakes fundamentally different? Alternatively, do they obey a universal law? In terms of kinematics, the empirical relationship among them strongly suggests the existence of a governing law. Over almost the entire range of natural and laboratory earthquakes, the seismic moment, M_0, and corner frequency (or cutoff frequency in engineering terminology), $f_c \sim 1/T$, show the following relationship: $M_0 \propto f_c^{-3}$[24, 27]. In other words, T must scale as $M_0 \propto T^3$[15]. Moreover, this relationship is observed not only in the final state but also in a temporal state as

$$M_0(t) \propto t^3 \tag{4}$$

at least for the Parkfield area in California[23] and Japan[20], which means that the system is self-similar[5]. If the rupture propagation velocity is almost constant and given as v_r, the current crack radius is written as $r = v_r t$, and the scaling relationship (4) suggests that the macroscopic energy release rate, $G := dM_0/dA$, of the system satisfies

$$G \propto F(v_r)\frac{dA^{2/3}}{dA} \propto F(v_r) r = v_r F(v_r) t, \tag{5}$$

where $A(\propto r^2)$ is the current area of the ruptured region. This proportionality (5) and the function F are simply derived from modeling the self-similar expansion of circular and elliptic singular cracks (Section 6.9 of Broberg[6]). In fact, a zeroth-order approximation of the earthquake faulting process has been provided by such a singular self-similar crack model.

However, a precise comparison of observations and the singular crack model reveals some discrepancies. In a traditional framework of linear elastic fracture mechanics, the energy release rate, G, must be balanced with the surface energy of the material, 2γ. If γ is a material constant, the relationship (5) never balances with 2γ unless the rupture propagation velocity asymptotes towards the Rayleigh wave speed, c_R, for mode-II ruptures or the shear wave speed, c_S, for mode-III ruptures, because $\lim_{v_r \to c_R, c_S} F(v_r) = 0$[6]. On the contrary, the typical rupture propagation velocity of earthquakes is $50-90$ % of the Rayleigh or shear wave speed of the Earth's crust (e.g.,[11, 18]). Thus, both the release and dissipation rate of energy must increase during dynamic rupture growth.

Another discrepancy is as follows. The singular crack model means that the slip velocity diverges in the vicinity of the rupture front due to the square-root singularity. This non-physical property must be somehow reduced by a dissipative process neglected in the Broberg relationship; however, a problem arises even in a non-singular crack model with a constant energy dissipation rate. According to numerical simulations with a friction model equivalent to such dissipation, the peak slip velocity appears an increasing function of r and t[4]. This tendency suggests that the peak slip velocity of $M9$ earthquakes (e.g., duration $T \sim 150$ s for the 2011 Tohoku earthquake) should be 10^4 times greater than that of $M1$ earthquakes (e.g., $T \sim 15$ ms for microearthquakes in a South African gold mine). However, seismic slip inversion analyses revealed that they are both on the same order of 1 m/s (e.g., Ide et al.[17] for $M9$; Yamada et al.[25] for $M1$). Therefore, a model of the dissipation process that restricts slip velocity for such a wide magnitude range is required to understand the physics of earthquakes.

In the following context, we show how dynamic rupture propagation can be modeled in earthquake mechanics. We note the following important properties that have been proposed by observational studies and should be modeled: 1) the rupture velocity is 50–90% of the Rayleigh or shear wave velocity and 2) the slip velocity on faults is finite and scale-invariant. Considering these properties, we focus on friction and off-fault damage to model the dissipation processes previously developed by multiple seismologists. A previous numerical study has already suggested that both friction and damage have a somewhat similar effect on the dissipation rate and self-similarity of rupture propagation[3, 4]. In this study, we analytically illustrate this similar effect by assuming self-similar crack growth.

2 Formulation and physically reasonable models

2.1 Self-similar displacement, velocity, and strain

First, we confirm the following property for a function of $X := x/t$, where $x \in \mathbb{R}^n$ ($n = 2$ or 3) and $t > 0$. Suppose that $\Phi(x,t) = \Phi(X)$ is compactly supported along x-axis, and $\operatorname{supp}\Phi := \{(x,t) \mid \Phi \neq 0\}$ is self-similarly growing inside of $\mathbb{R}^n$. Then,

$$\int_{\mathbb{R}^n} \Phi(X) dx = t^n \int_{\operatorname{supp}\Phi(X)} \Phi(X)\, dX \tag{6}$$

where t^n comes from Jacobian $|\nabla X|^{-1}$, and the integral with respect to X is independent of t. Via time-derivative, the following also holds:

$$\int_{\mathbb{R}^n} \partial_t \Phi(X)\, dx = t^{n-1} \int_{\operatorname{supp}\Phi(X)} \Phi(X)\, dX. \tag{7}$$

A complete partial differential equation was given as an initial and boundary value problem for the case of an arbitrary-shaped fault surface embedded in a finite linear elastic body[13]. In this research, we simply consider that an already ruptured region $\Gamma(t) = \operatorname{supp} D$ at time t is a simply connected subset of a planar fault $\left(\subset \mathbb{R}^2\right)$ embedded in an infinite homogeneous elastic and/or inelastic domain $\left(= \mathbb{R}^3\right)$.

Hereafter, we assume a self-similar system that satisfies eq.(4). The self-similarity can be represented by a homogeneous function of degree N:

$$u(\alpha x, \alpha t) = \alpha^N u(x,t), \tag{8}$$

for an arbitrary positive constant, α, which is satisfied if

$$u(x,t) = t^N u(X) \tag{9}$$

holds, where $X := x/t$ again. From the definition of slip, (1),

$$D(x,t) = t^N D(X) \tag{10}$$

also holds, and substituting it into eq.(3) and eq.(6) results in

$$M_0(t) = \mu t^N \int_\Gamma D(X)\, dx \propto t^{N+2}.$$

Therefore, through a comparison of this relationship with the empirical law (4), we conclude that $N = 1$.

The spatial- and time-derivatives of eq. (8) yield that strain $\varepsilon := \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$ and velocity $v := \partial_t u$ are homogeneous functions of degree $N - 1 = 0$. Hence,

$$v(x,t) = v(X), \tag{11}$$
$$\varepsilon(x,t) = \varepsilon(X) \tag{12}$$

are obtained. Specifically, eq.(11) indicates the scale invariance of peak slip velocity mentioned in the Introduction if v is finite because of some dissipative process.

2.2 Linear PDE and Reasonable Modeling of Friction

Here, we consider the requirement to reproduce self-similar and scale-invariant peak slip velocity. To avoid the emergence of infinite stress, strain, and velocity in the vicinity of a dynamically propagating rupture front in numerical models, various types of on-fault frictional property have been proposed. Particularly for the co-seismic slip velocity range ($\sim$ 1 m/s), a slip-weakening law

$$f(x,t) = (f_s - f_d)\,\phi\left(\frac{|D(x,t)|}{D_c}\right) + f_d \tag{13}$$

has been widely employed, where f_s and f_d are the static and dynamic friction, respectively, D_c is the characteristic slip distance, $\phi(s) := (1-s)H(1-s)$, and $H(\cdot)$ is the Heaviside function (e.g.,[16, 2]). This model is approximately consistent with results from laboratory stick-slip experiments of rock samples above $\sim$ 1 cm/s (e.g., [21]). If eq.(10) holds with $N = 1$, slip-dependent friction is represented as

$$f(x,t) = f(tD(X)). \tag{14}$$

Also, the fracture energy (i.e., energy dissipation during rupture growth of unit area) is constant under the model (13) because the energy is approximated as $\frac{1}{2}(f_s - f_d)D_c$[22]. In the following, we show that dynamic crack growth under the slip-weakening friction law will not be self-similar as in eqs.(8) and (11).

We consider linear elasticity, where the stress change σ and strain are linearly connected via the elasticity tensor C as follows

$$\sigma(x,t) = \sigma(X) = C\varepsilon(X).$$

We should assume that v and σ are zero for $t \leq 0$ because velocity and stress perturbations are only caused by an earthquake occurring at $t > 0$. Therefore, the governing equation is the following boundary value problem with respect to X:

$$\begin{cases} \rho\partial_t v(X) = \nabla\cdot\sigma(X), & X \in \mathbb{R}^3\backslash\Gamma \\ \partial_t\sigma(X) = C\partial_t\varepsilon(X), & X \in \mathbb{R}^3\backslash\Gamma \\ \sigma(X)\nu^\Gamma = f, & X \in \Gamma \\ v \to 0, \sigma \to 0, & |X| \to \infty \text{ (i.e., } |x| \to \infty \text{ or } t \downarrow 0) \end{cases} \tag{15}$$

where ρ is the material density and $\sigma\nu^{\Gamma}$ is the traction change on Γ due to the faulting process. In our model, the fault is a planar shear crack (i.e., $D\cdot\nu^{\Gamma}=0$ and ν^{Γ} is constant), which does not change the normal stress on Γ. Therefore, $\sigma\nu^{\Gamma}$ is parallel to Γ (i.e., $\left(\sigma\nu^{\Gamma}\right)\cdot\nu^{\Gamma}=0$) and balances with the decrease in friction on the sliding surface f.

So that eq.(15) represents a self-similar system, $f=f(X)$ must hold, which tells us that any slip-dependent friction (14) cannot completely reproduce self-similar crack growth. Even if eq.(13) is introduced, f becomes almost constant for a sufficiently larger value of t because of eq.(14). This means that the problem asymptotes towards the singular crack problem, in which slip velocity diverges. In the same way, the time-weakening friction, $f\propto\phi((t-t_x)/t_c)$, where t_x is the time at which point $x\in\Gamma$ is ruptured and t_c is a characteristic time[3], is also inappropriate for our purpose. One possibility is a velocity-dependent friction $f(V(X))$. The above analysis shows that, for example, velocity-weakening[8] and velocity-strengthening[3] friction models are reasonable for reproducing the scale-invariant peak slip velocity.

2.3 Energy Conservation Law with a Non-linear Damage Model

Many geological investigations have reported highly damaged rock surrounding faults with observations of an enormous number of microcracks in fault outcrops (e.g.,[7, 10]). The microcrack density per unit area of a cross section decays exponentially as the distance from principal slip plane increases[10]. Therefore, intense co-seismic stress concentration around the dynamic rupture front could generate these microcracks, which could dissipate a non-negligible amount of energy by creating new micro-surfaces. Theoretical modeling consisting of linear elasticity and on-fault friction, as discussed in the previous subsection, has played a crucial role in understanding fault dynamics. However, modeling the damage generation is necessary if it dominates during an actual energy dissipation process.

Several theoretical and numerical models of dynamic damage generation have been proposed. As a realistic model, Yamashita[26] considered each opening microcrack distributed around the fault and executed a finite difference calculation. On the other hand, continuum mechanics-based modeling has been developed by many authors because such discrete modeling is not easy to handle. In the following, we considier a constitutive law including plasticity, which is described as follows:

$$\partial_t\sigma = C\left(\partial_t\varepsilon-\partial_t\varepsilon^p\right), \tag{16}$$

where the plastic strain ε^p is zero in the early stage of deformation (i.e., small value of σ) and its rate $\partial_t\varepsilon^p$ depends non-linearly on σ; see Dunham[9] for a specific case of the evolution law of ε^p. By introducing elastic and plastic parts of stress σ as $\sigma^e := C\varepsilon$ and $\sigma^p := C\varepsilon^p$, eq.(16) can be written as

$$\partial_t\sigma = \partial_t\left(\sigma^e-\sigma^p\right). \tag{17}$$

Laboratory triaxial compression experiments using rock samples have shown that the stress accumulation rate is almost constant during the early stage of loading and decreases in a highly stressed state, as with eq.(17). Immediately after the experiment, CT imaging revealed an enormous amount of microcracks inside the sample[19]. Therefore, eq.(17) applies to damage modeling in the highly compressional state.

Here, we discuss a macroscopic energy conservation law including off-fault plastic deformation and on-fault friction. By taking an inner product of velocity, v, and the equation of motion, $\rho \partial_t v = \nabla \cdot \sigma$, we get

$$\frac{\rho}{2} \int_{\mathbb{R}^3 \setminus \Gamma} \partial_t |v|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^3 \setminus \Gamma} \partial_t \operatorname{tr}(\sigma^e \varepsilon) \, dx = \int_\Gamma V \cdot f \, dx + \int_{\mathbb{R}^3 \setminus \Gamma} \operatorname{tr}(\sigma^p \partial_t \varepsilon) \, dx, \tag{18}$$

where the boundedness of v, σ, and ε is assumed on the basis of a physical requirement. See Hirano[14] for a detailed derivation; i.e., eq.(18) can be obtained by substituting eq.(17) into eq.(3.1) of [14] although the original equation was based on linear elasticity. On the left-hand side, the first and second terms represent the rate of bulk kinetic energy and released bulk elastic strain energy, respectively. On the right-hand side, the first term is the frictional work rate, which represents dissipated energy due to friction, and the second term is similar to the virtual work rate due to the plastic part of the stress. Thus, this second term refers to energy dissipation due to plastic strain or damage.

3 Discussion and Conclusion: Macroscopic Equivalence of Friction and Damage

Both the non-linear model with off-fault damage and the linear elastic model with on-fault friction have contributed to our understanding of self-similar dynamic rupture growth. In this section, we confirm that the energies of these two different models are macroscopically equivalent only if $f = f(X)$ via a dimensional analysis. As with above, v, ε, σ^e, σ^p, and V are assumed to be a function of X for scale invariance.

Given eqs.(6) and (7), each term in eq.(18) is proportional to t^2 if $f = f(X)$ holds. Otherwise ($f \neq f(X)$), the first term in the right-hand side is not proportional to t^2, which means that friction does not contribute to the energy dissipation for a sufficiently large value of t. Eq.(18) also means that the total released potential energy is proportional to t^3 according to time integration, which was observed seismologically as eq.(4). Hence, we can conclude the following model patterns:

1. With linear elasticity and any self-similar friction, $f(X)$ (e.g.,[8]), the energy balance holds even though the second term of the right-hand side of eq.(18) is absent.

2. With linear elasticity and slip- or time-weakening friction (e.g.,[2, 3, 4]), self-similarity is achieved not in the strict sense but as an asymptote. Here, the energy dissipation rate is not proportional to t^2, which implies the divergence of the slip velocity as an analogy of the singular crack problem.
3. With the damage model and self-similar friction (e.g.,[9]), both show energy dissipation propotional to t^2 and, therefore, contribute to the energy balance.
4. With the damage model and non-self-similar friction (e.g.,[4]), only damage becomes a dominant dissipative process to satisfy the energy balance and self-similarity.

In the above, the cited authors executed 2-D numerical simulations using each type of model, while our analytical modeling is valid also for 3-D cases. A key finding is that patterns 1, 3, and 4 will contribute to reproducing the sub-Rayleigh rupture velocity and scale-invariant peak slip velocity. This conclusion means that we can mimic damage rheology by considering an appropriate friction model, as conducted numerically[4]. On the other hand, seismological observations cannot conclude whether damage or friction is more important for fault mechanics because observations are typically conducted at far-field. Hence, seismological and geological models can be independent, where the former can be achieved even under model pattern 1, while the latter requires model patterns 3 or 4.

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