AGU 2024
Dec. 12
[S41G-3390]
Poster Downloadable

Physics of a stochastic partial differential equation for kinematic rupture modeling

Shiro Hirano (Hirosaki Univ., E-mail: hirano@hirosaki-u.ac.jp, Web: https://interfacial.jp/)
Hirosaki university

1. Introduction

Inverted fault motions are diverse (Fig.1), but statistically show some empirical laws.

This study aims to explore a stochastic model to represent empirical laws, which may contribute to the understanding of fault mechanics/stochastic ground motion simulation.

Abbreviation

- $\boldsymbol{x}$: Position on a fault plane Γ ($\in \mathbb{R}^n$, $n = 0, 1, 2$)
- $D(\boldsymbol{x}, t)$: Slip distribution along Γ
- $V(\boldsymbol{x}, t) := \dot{D}$: Slip rate distribution
- $S(t) := \int_\Gamma V(\boldsymbol{x}, t)\,\mathrm{d}\boldsymbol{x}$: Source time function (STF) $\propto$ moment rate function
- T: Duration of slip (i.e., $T < t \Rightarrow V \equiv 0$)

Point source ($n = 0$)

The STFs follow:

1. Non-negative, compactly supported, and dominantly unimodal [Yin+ 2021, *SRL*].
2. ω^{-2}-model: Fourier amplitude follows a power law, i.e., $|\mathfrak{F}S(f)| \sim f^{-2}$ for $\frac{1}{T} \ll f$ [Boatwright 1980, *BSSA*; Abercrombie 1995, *JGR*].

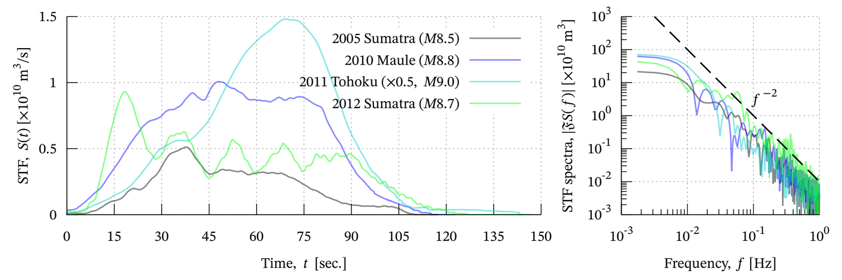

Fig.1 Various STFs and their spectra (data from SCARDEC).

3. **The cube law**: $\int_0^t S(\tau)\,\mathrm{d}\tau \sim t^3$ ($t \ll T$), [Uchide & Ide 2010, *JGR*; Meier+ 2016, *GRL*]
   and $M_0 := \mu \int_0^T S(\tau)\,\mathrm{d}\tau \propto T^3$ [Houston 2001, *JGR*]
4. **The GR law**: The probability density of M_0 is approximated as $P(M_0) \propto M_0^{-\frac{2}{3}} \propto T^{-2}$ [Gutenberg & Richter 1944, *BSSA*].

The above four laws are satisfied [Hirano 2022, *Sci.Rep.*] if

$$S(t) = \int_0^t X_s^{(1)} X_{t-s}^{(2)}\,\mathrm{d}s, \quad (1)$$

where $X_t^{(1)}$ and $X_t^{(2)}$ are the Bessel processes, solutions of a stochastic differential equation:

$$\mathrm{d}X_t = \frac{(3-4b)\,\mathrm{d}t}{2X_t} + \mathrm{d}B_t \quad (2)$$

Fig.2: Two Bessel processes, solutions of eq.(2)

(Fig.2&3), where b is the b-value of the GR law, and B_t is a standard Brownian noise.

Finite fault ($n = 2$)

5. k^{-2}-model: Spectral fall-off rate of Fourier amplitude of the final slip in the wavenumber domain is -2; that is

$$\left| \int_{\mathbb{R}^2} D(\boldsymbol{x}, T)\, e^{2\pi i \boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x} \right| \sim |\boldsymbol{k}|^{-2},$$

where $\boldsymbol{k}$ $\left(\in \mathbb{R}^2\right)$ is the wavenumber vector [Herrero & Bernard 1994, *BSSA*].

6. Ballistic rupture propagation slightly slower than the wave speed.

Mathematical model for all (six) empilical laws?

Spatial distribution $\Rightarrow$ Stochastic Partial Differential Equation (SPDE) for D is required.

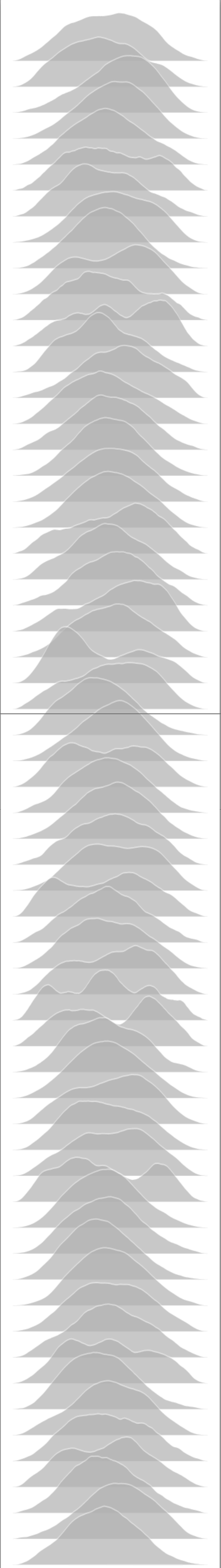

Fig.3: Synthetic $S(t)$ by eqs.(1) and (2).

2. Mathematical model extension

Starting from the Bessel process:

$$\mathrm{d}X = \frac{\mu - 1}{2}\frac{\mathrm{d}t}{X} + \sigma\,\mathrm{d}B.$$

Can we replace $\mathrm{d}B$ by a space-time white noise?

NO! because it generates noise everywhere (i.e., even currently non-slipping region during the earthquake may slip due to the noise).

The squared Bessel process

$V := X^2$ and Itô's lemma yields the following:

$$\dot{V} = \mu + 2\sigma\sqrt{V}\,\dot{B},$$

which generates the noise only at $V > 0$.

Rupture (or energy) propagation in space

We consider the following system:

$$\dot{D} = V_+ = \frac{1}{2}\left(V + |V|\right), \quad (3)$$

$$\dot{V} = \Delta D + \mu + 2\sigma\sqrt{V_+}\,\dot{B} + \delta(\boldsymbol{x})\,H(t), \quad (4)$$

where V_+ is non-negative slip rate, and $\delta(\boldsymbol{x})$ is Dirac's δ-function for ignition.

If $\sigma = 0$, the above is a wave equation (Figs.4&5).

$\because$ Differentiate eq.(4), then substitute eq.(3) into it.

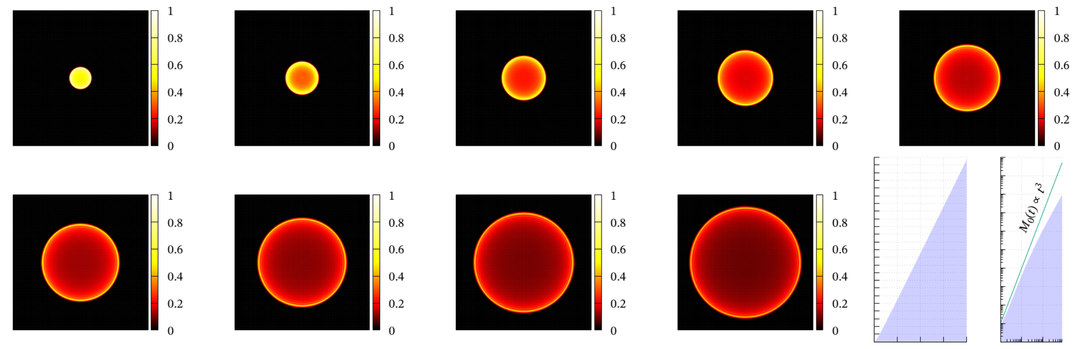

Fig.4: Solution V_+ for a 2-D wave equation ($\mu = \sigma = 0$).

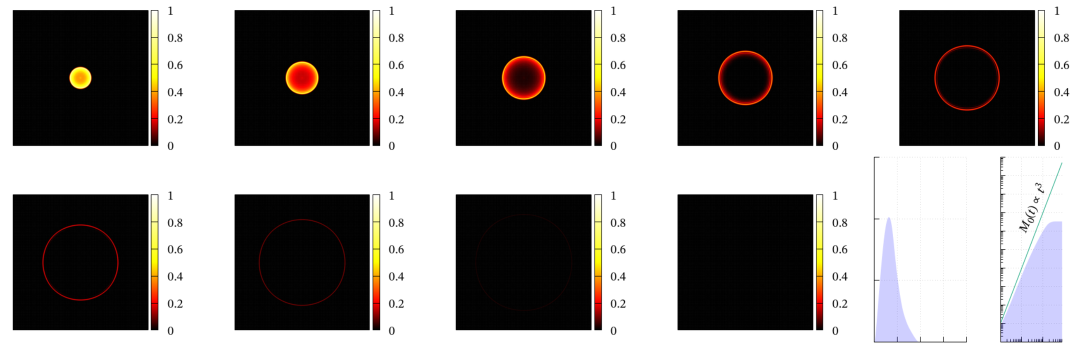

Fig.5: Solution V_+ with constant friction ($\mu < 0, \sigma = 0$).

If $\sigma \neq 0$

The SPDE is accompanied by noise only at the slipping region ($V > 0$), which may reflect hight-frequency radiation due to fault roughness and/or scattering by the surrounding region including fault damage zone (Fig.6).

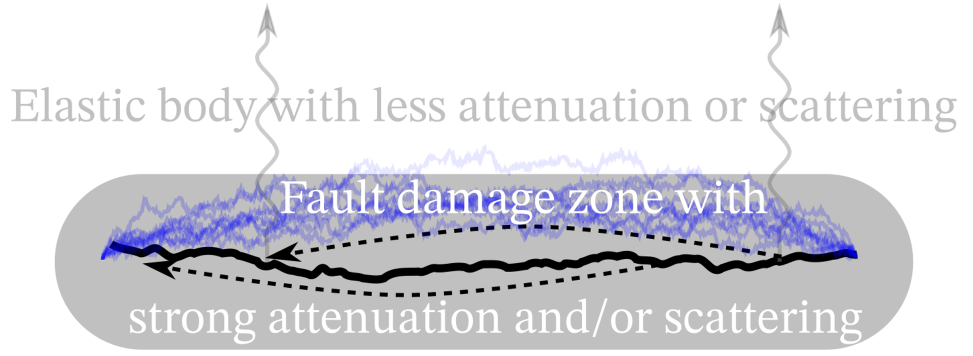

Fig.6: Schematic illustration of the noise term.

3. Numercal solutions for 1-D case

We solved the system numerically (Finite difference, Euler-Maruyama scheme) with $\sigma = 1$.

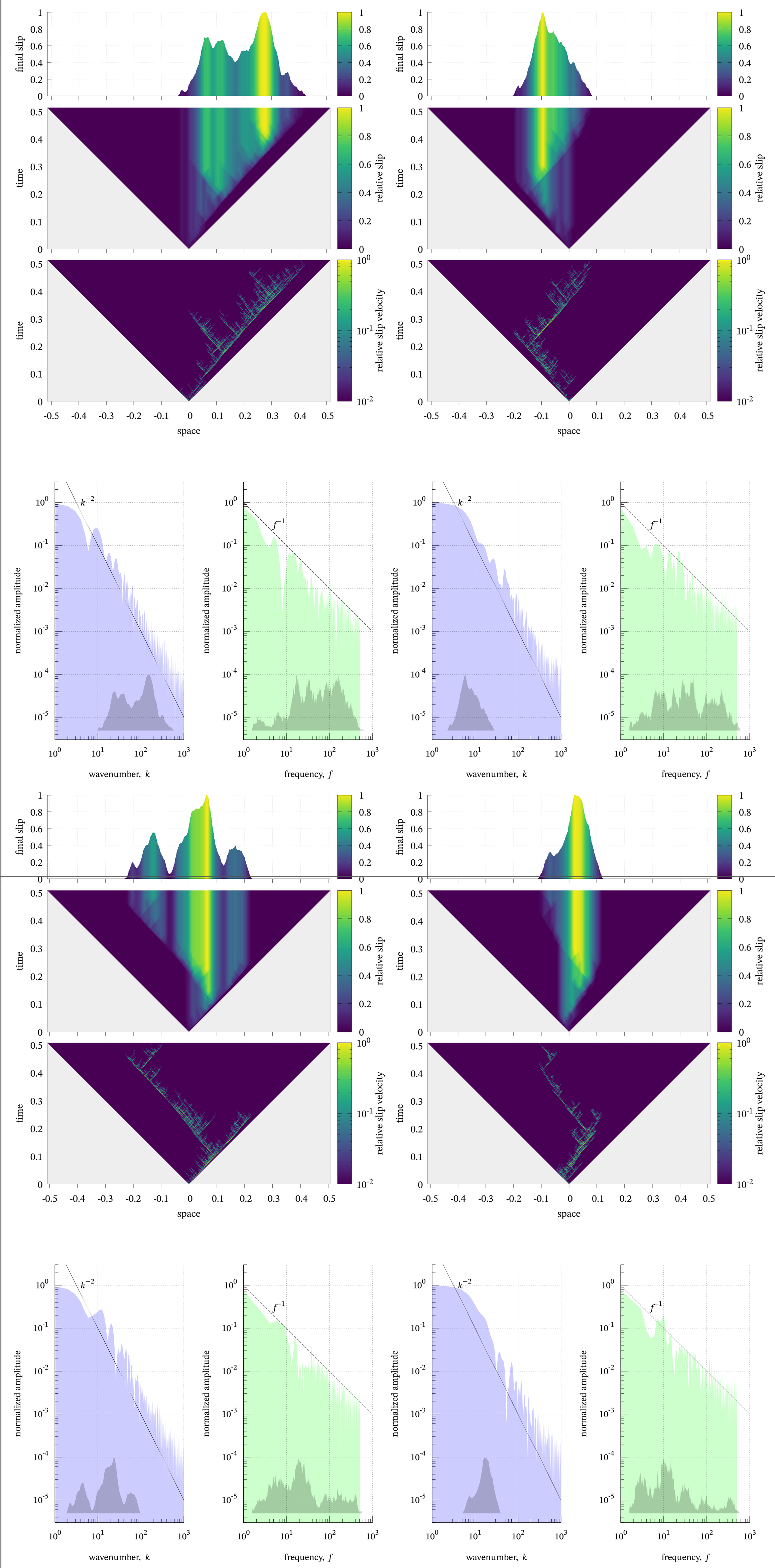

Fig.7: Final slip (top), slip evolution (middle triangle), and slip rate (bottom triangle) following the stochastic 1-D model with $\mu = -3$. The four solutions died at $t \sim 0.5$, but the durations are determined by the stochastic process. Spectra are for the final slip in the wavenumber domain (blue) and the STF in the frequency domain (green).

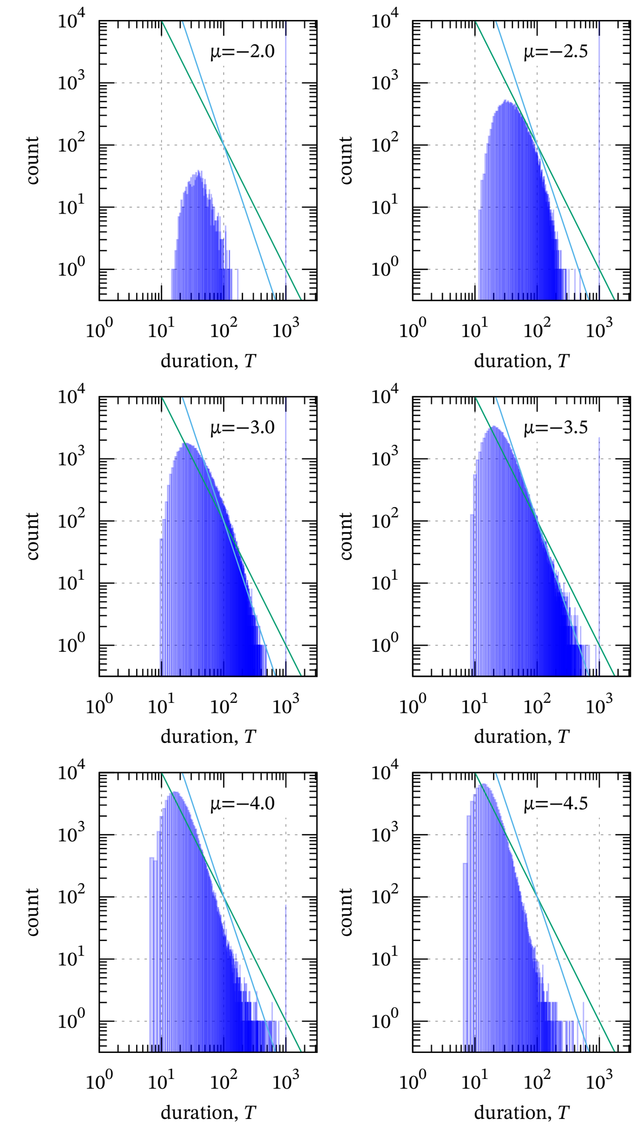

Fig.8: Histogram of event duration for various values of friction μ. Stronger friction ($\mu < -3$) seems to yields a power law. Lines indicate $N \propto T^{-2}$ and T^{-3}.

Cases for $\mu = -3$ show various slip evolution (Fig.7).

Spontaneous termination in many cases.

$\because$ The friction term μ always tranquilizes the solution, while the noise term $2\sqrt{V_+}\dot{B}$ strongly perturbs during a high slip rate. The termination follows stochastically.

Heterogeneous final slip distributions following k^{-2}-type spectral fall-off rate in the wavenumber domain (blue in Fig.6).

$\because$ After termination of slip ($t \to \infty$), eq.(4) becomes

$$-\Delta D = \delta(x) \Leftrightarrow (4\pi k)^2 \widehat{D} = 1,$$

where $\widehat{D}$ is the Fourier transform of D, and k is the 1-d wavenumber. Hence $\widehat{D} \propto k^{-2}$.

Predominantly unilateral propagation slightly slower than the wave speed (edges of gray triangles).

$\because$ Sometimes, the front cannot propagate stochastically due to the noise, which delays the macroscopic propagation speed.

Frequent backward propagation like *boomerang rupture* [Hicks+ 2020, *Nat.Geos.*].

ω^{-1}-type spectra (green in Fig.6).

$\because$ After integration w.r.t. space ($\Delta \mapsto 0$, $\delta \mapsto 1$), the solution is essentially Brownian motion.

Power law for event duration (Fig.8)

$N \propto T^{-3}$ for $\mu < -3$.

Shorter events dominate comparing to the GR law ($N \propto T^{-2}$) because of the dimension ($n = 1$)?

In 1-D, $M_0 \propto T^2$, implying that $P(M_0) \propto T^{-\frac{4}{3}}$.

4. Numerical solutions for 2-D case

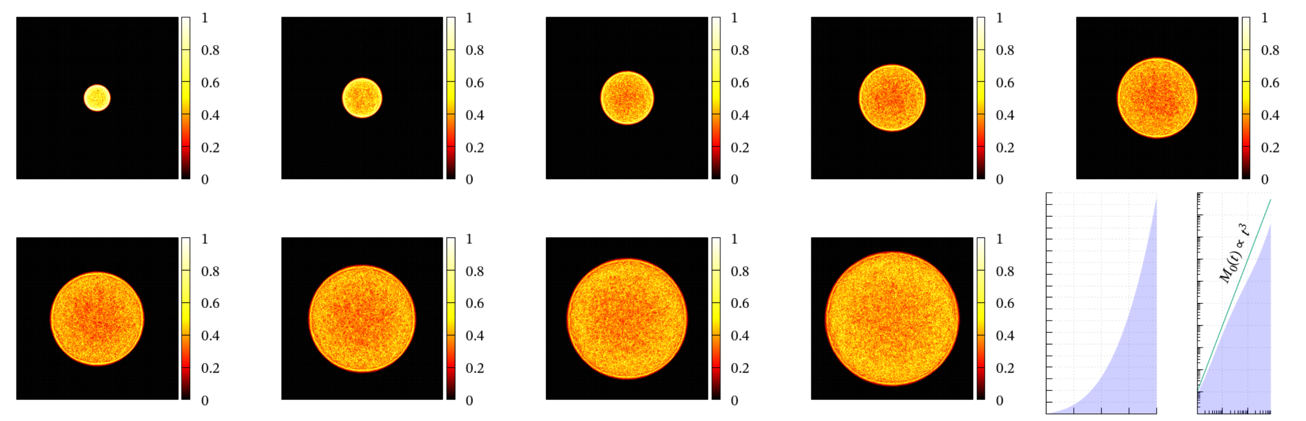

Fig.9: Evolution of V with $\mu = 0$ and $\sigma = 1$.

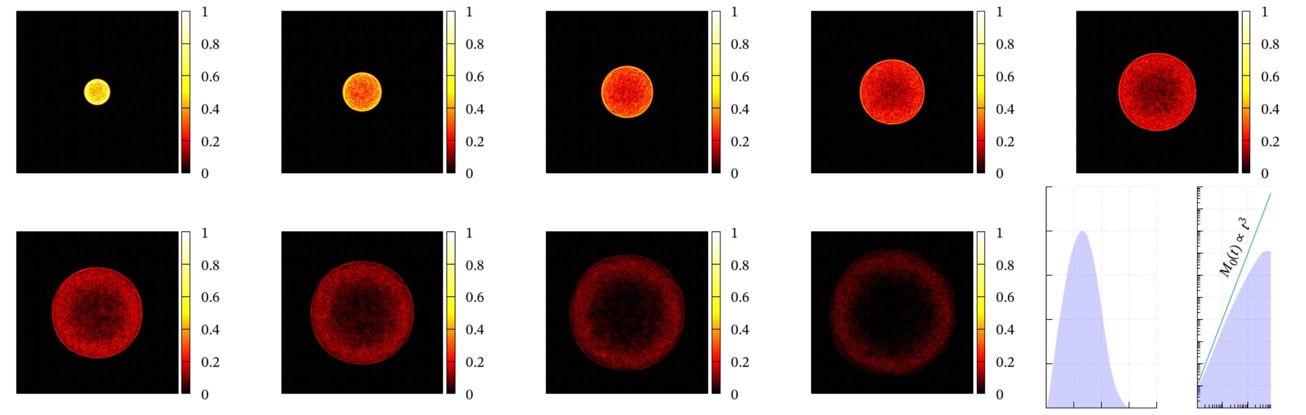

Fig.10: Evolution of V with $\mu = -3$ and $\sigma = 1$.

Frictionless case (Fig.9)

The slip rate diverges.

Frictional case (Fig.10)

The slip rate terminates spontaneously.

ω^{-2}-model and the cube law are satisfied.

No macroscopic heterogeneity.

No variation in event size.

Why doesn't it work?

Integrating the SPDE over currently ruptured region $A(t)$ for $t > 0$:

$$\int_{A(t)} \dot{V}\,\mathrm{d}\boldsymbol{x} = \dot{S},$$

$$\int_{A(t)} \left(\Delta D + \mu + 2\sigma\sqrt{V_+}\,\dot{B} + \delta(\boldsymbol{x})\,H(t)\right)\mathrm{d}\boldsymbol{x} = \mu\,A(t) + 1,$$

meaning that amplitude of deceleration is proportional to the current area $A(t)$.

For the GR law, the probability of rupture termination in each time step must be proportional to $\frac{1}{A(t)}$ [Scholz 1986, *BSSA*].

The friction term must have some different structure(?)

5. Conclusions

Stochastic kinematic source modeling

Many empirical laws should be satisfied statistically.

Potentially contributes to fault dynamics/strong ground motion simulation.

The point source model works well [Hirano 2022 *Sci.Rep.*].

The 1-D model satisfies some of the laws, except ω^{-2}-type spectrum and the GR law.

The 2-D model does not show diversity: almost deterministic and characteristic functions.

Future work?

Modification of the 2-D model that satisfies all the empirical laws and coincides with the point source model after integration w.r.t. space.

Acknowledgment

This work has been supported by JSPS KAKENHI 22K03782. The source time functions in Fig.1 are from the SCARDEC database http://scardec.projects.sismo.ipgp.fr/